Stochastic Processes and their Applications 19 (1985) 305-314
North-Holland

ON EFFICIENT STOPPING TIMES

Valeri T. STEFANOV

Institute of Mathematics, Bulgarian Academy of Sciences, 1090 — Sofia, P.O. Box 373, Bulgaria

Received 5 January 1984
Revised 20 March 1984

This paper is devoted to efficient sequential estimation in stochastic processes whose corresponding sufficient statistics are processes with stationary independent increments. It is proved that a stopping time is efficient if and only if it represents a time of the first attaining of a hyperplane., which cannot 'be passed', in the sense which is made precise below. The problem of determining the explicit form of the hyperplanes which cannot 'be passed' is also discussed.

sequential plan * Cramér—Rao inequality * efficient Markov stopping time

1. Introduction

Recently, many papers dealing with the problem of efficient sequential estimation, in the Cramér–Rao sense, have appeared [7, 8, 15, 17, 18, 20, 23, 24, 25]. Also if a stopping time is efficient, then the measure generated by the sufficient statistic is accumulated on some hyperplane. An interesting problem is the following: which hyperplanes are connected with efficient stopping times? (see section 'Unsolved problems' in [13]). We consider the case when the sufficient statistic is a process with stationary independent increments. Under mild conditions, W. Winkler and J. Franz [24] have shown that a process with stationary independent increments forms a sufficient statistic if and only if it belongs to the exponential class of processes. Thus we limit our consideration to this class.

Some preliminary results and basic definitions are given in Section 2. In Section 3 we prove a necessary and sufficient condition for a stopping time to be efficient and in Section 4 we obtain useful expressions for determining the explicit form of the efficient stopping times. Some examples are also discussed.

2. Preliminaries

Let $\{X(t)\}_{t\in T}$ be a random process defined on the probability space $(\Omega, \mathscr{B}, P_\theta)$ with values in $(\mathbb{R}^n, \mathscr{B}_{\mathbb{R}^n})$, where θ is a parameter, $\theta\in\Theta\subset\mathbb{R}^k$. The time parameter t may be discrete ($T=\{0, 1, 2, \ldots\}$) or continuous ($T=[0, +\infty)$). Let the process $\{S(t)\}_{t\in T}$ be a sufficient statistic for θ ([2]). We observe the process $\{S(t)\}_{t\in T}$ up to

0304-4149/85/$3.30

some Markov stopping time τ and we want to estimate a given real function $h(\theta)$. As a measure of efficiency we shall use the lower bound of the Cramér-Rao inequality in the sequential case [14, 23, 24]. A result of Döhler [6], which asserts that $(S(\tau), \tau)$ forms a sufficient statistic in the sequential case, allows us to consider only estimators which are real functions of $(S(\tau), \tau)$. In the present paper we assume that $\{S(t)\}_{t\in T}$ belongs to the exponential class of processes, defined as follows:

Definition 1 [24]. The random process $\{S(t)\}_{t\in T}$ belongs to the exponential class if the following conditions are fulfilled:

(i) $\{S(t)\}_{t\in T}$ is continuous in probability, has stationary independent increments and $S(0)=0$ P_θ-a.s. for each $\theta\in\Theta$.

(ii) The probability distributions at time t are dominated by a σ-finite measure ν and the densities with respect to ν may be represented in the form

$$f(x, t, \theta)=g(x, t)\exp\left(\sum_{i=1}^{m} \alpha_i(\theta)x_i+\beta(\theta)t\right)$$

where $x=(x_1, \ldots, x_m)\in\mathbb{R}^m$, g is a nonnegative function defined on $\mathbb{R}^m\times T$ and α_1, $\alpha_2, \ldots, \alpha_m$, β are nonconstant functions defined on Θ.

Following [24] we can properly treat the problem of efficient estimation if we additionally assume that $\kappa=m$, $\alpha_1, \ldots, \alpha_m$, β are continuously differentiable and M^{-1} exists, where M denotes the matrix $(\partial\alpha_i(\theta)/\partial\theta_j)_{i,j}$. However, these conditions allow us to introduce a canonical parametric representation of our exponential family. Thus without loss of generality we shall suppose throughout the paper that $\alpha_i(\theta)=\theta_i$. We remark that the sufficient statistics of the most important processes occurring in practice belong to the exponential class of processes and satisfy the conditions mentioned above (binomial, negative-binomial, gamma, Poisson, Wiener, multinomial, compound Poisson with jumps of exponential type processes etc.).

Definition 2. We call a sequential plan the triplet (τ, f, h) consisting of a Markov stopping time τ with respect to $\{\mathscr{F}_t\}_{t>0}$, $\mathscr{F}_t:=\sigma(X(s){:}\ s\leq t)$, a real function $h=h(\theta)$ and an unbiased estimator $f=f(S(\tau), \tau)$ of h. We shall consider only finite stopping times i.e. $P_\theta\{\tau<+\infty\}=1$, for each θ from some m-dimensional interval I, $I\subset\Theta$. From the well known lemma of Sudakov [2, 21] we have for every finite stopping time τ that for each θ, $\theta_0\in I$:

$$P_\theta\{(S(\tau), \tau)\in B\}=\int_{\{(S(\tau),\tau)\in B\}}\exp\left(\sum_{i=1}^{m}\theta_iS_i(\tau)+\beta(\theta)\tau\right)\mathrm{d}P_{\theta_0},\quad B\in\mathscr{B}_{\mathbb{R}^m}.$$

The conditions, written below are known as Cramér-Rao regularity conditions([23]):

$$\frac{\partial}{\partial\theta_i}\int\exp\left(\sum_{i=1}^{m}\theta_iS_i(\tau)+\beta(\theta)\tau\right)\mathrm{d}P_{\theta_0}=\int\frac{\partial}{\partial\theta_i}\exp\left(\sum_{i=1}^{m}\theta_iS_i(\tau)+\beta(\theta)\tau\right)\mathrm{d}P_{\theta_0},$$

$$\frac{\partial}{\partial\theta_i}\int f(S(\tau),\tau)\exp(\ldots)\,\mathrm{d}P_{\theta_0}=\int\frac{\partial}{\partial\theta_i}f(S(\tau),\tau)\exp(\ldots)\,\mathrm{d}P_{\theta_0},$$

$$E_\theta\left(S_i(\tau)+\frac{\partial\beta}{\partial\theta_i}\tau\right)^2<+\infty,\quad i=1,2,\ldots,m,\ \theta\in I.$$

Under these conditions we obtain the Cramér–Rao inequality [23, 24] for the variance of $f(S(\tau),\tau)$, where equality holds if and only if for each $\theta\in I$ P_θ-a.s.

$$f(S(\tau),\tau)=\sum_{i=1}^{m}b_i\left(S_i(\tau)+\frac{\partial\beta}{\partial\theta_i}\tau\right)+b_{m+1}\tag{1}$$

where $b_1, b_2,\ldots,b_{m+1}$ are real numbers which may depend on θ.

Remark 1. It is easy to see that a sufficient condition for the Cramér–Rao regularity conditions to be satisfied is the finiteness of the first two moments of the sufficient statistic i.e.

$$E_\theta S_i^2(\tau)<+\infty,\quad E_\theta\tau^2<+\infty,\qquad i=1,2,\ldots,m,\ \theta\in I.\tag{2}$$

The definitions given below are basic in efficient sequential estimation theory. Some of them, in one-dimensional case, may be found in the book [2].

Definition 3. The sequential plan (τ,f,h) is said to be efficient if there exists a m-dimensional interval I, $I\subset\Theta$ such that, for each $\theta\in I$ equality holds in the Cramér–Rao inequality. The estimator f and function h are then called efficient and efficiently estimable respectively.

Definition 4. The Markov stopping time τ is said to be efficient if there exist h and f, such that (τ,f,h) is an efficient sequential plan.

Definition 5. The Markov stopping time τ is said to be efficient for Θ_0, $\Theta_0\subset\Theta$, if there exist h and f, such that equality holds in the Cramér–Rao inequality for each $\theta\in\Theta_0$, for the sequential plan (τ,f,h).

It is well known [24] that for every efficient stopping time τ the equality

$$\sum_{i=1}^{m}a_iS_i(\tau)+a_{m+1}\tau=a_{m+2}\tag{3}$$

holds P_θ-a.s. for each $\theta\in I$, where $a_1, a_2,\ldots,a_{m+2}$ are real numbers such that $a_{m+2}\neq 0$, $\sum_{i=1}^{m+1}a_i^2\neq 0$. Without loss of generality we assume that $a_{m+2}>0$. In other words, the equality (3) means that the measure generated by the sufficient statistic $(S(\tau),\tau)$ is accumulated on some hyperplane A

$$A:=\left\{(x_1,\ldots,x_{m+1})\in R^{m+1}:\ \sum_{i=1}^{m+1}a_ix_i=a_{m+2}\right\}$$

where $a_{m+2}>0$ and $\sum_{i=1}^{m+1} a_i^2 \neq 0$. Define

$$Z(t) := \sum_{i=1}^{m} a_i S_i(t) + a_{m+1} t,$$

$$\tau_A := \inf\{t>0: Z(t) = a_{m+2}\}, \qquad \eta_s := \inf\{t>0: Z(t) \geq s\}, \quad s>0.$$

Let τ be an efficient stopping time. Thus $E_\theta \tau < +\infty$. By (3) we also have $E_\theta Z(\tau) = a_{m+2}$. So ([4, 12])

$$E_\theta Z(\tau) = E_\theta \tau E_\theta Z(1)$$

for each θ, for which τ is efficient. But $E_\theta Z(\tau)>0$ and $E_\theta \tau > 0$. Thus the largest subset for which τ may be efficient is the following:

$$\Theta_1 := \{\theta \in \Theta: E_\theta Z(1) > 0\}.$$

Definition 6. We say that the hyperplane A can 'be passed' by the process $\{S(t), t\}_{t\in T}$ if, for each $\theta \in \Theta_1$,

$$P_\theta\{Z(\eta_{a_{m+2}}) > a_{m+2}\} > 0. \tag{4}$$

In fact, this definition is more natural than the definition of 'passed hyperplanes' in the author's previous paper [20]. Denote by $\mathcal{K}$ the family of all hyperplanes which cannot 'be passed' by the process $\{S(t), t\}_{t\in T}$ and let $W := \{\tau_A: A \in \mathcal{K}\}$.

Remark 2. Since $E_\theta Z(1) > 0$, $\theta \in \Theta_1$, we have [9, p. 494]

$$P_\theta\{\eta_{a_{m+2}} < +\infty\} \geq P_\theta\{\sup_{0<t<+\infty} Z(t) = +\infty\} = 1, \quad \theta \in \Theta_1.$$

From the finiteness of $\eta_{a_{m+2}}$ and the lemma of Sudakov, we have for each $\theta_1, \theta_2 \in \Theta_1$, that $P_{\theta_1,\tau}$, $P_{\theta_2,\tau}$ are mutually absolutely continuous, where $P_{\theta,\tau}$ means the restriction of P_θ to the σ-algebra generated by $(S(\tau), \tau)$. Thus the equality $P_{\theta_0}\{Z(\eta_{a_{m+2}}) > a_{m+2}\} = 0$ for some $\theta_0 \in \Theta_1$, yields $P_\theta\{Z(\eta_{a_{m+2}}) > a_{m+2}\} = 0$ for each $\theta \in \Theta_1$. This means that $A \in \mathcal{K}$ if and only if $P_\theta\{Z(\eta_{a_{m+2}}) > a_{m+2}\} = 0$ for each $\theta \in \Theta_1$. Thus $\tau_A = \eta_{a_{m+2}}$ P_θ-a.s. $\theta \in \Theta_1$, when $\tau_A \in W$.

3. Main result

Theorem 1. *A stopping time τ is efficient if and only if τ belongs to W. Every τ_A which belongs to W is efficient for Θ_1. A real function $f = f(S(\tau), \tau)$, $\tau \in W$, is an efficient estimator if and only if $f(S(\tau), \tau)$ is a linear function of $S_1(\tau), S_2(\tau), \ldots, S_m(\tau), \tau$. All efficiently estimable functions satisfy the condition*:

$$E_\theta\left(\sum_{i=1}^{m} c_i S_i(\tau_A) + c_{m+1}\tau_A + c_{m+2}\right) = \frac{\alpha_{m+2}\left(\sum_{i=1}^{m} - c_i \frac{\partial \beta}{\partial \theta_i} + c_{m+1}\right)}{a_{m+1} - \sum_{i=1}^{m} a_i \frac{\partial \beta}{\partial \theta_i}} + c_{m+2}$$

where $c_1, c_2, \ldots, c_{m+2}$ *are arbitrary real numbers and* $a_1, a_2, \ldots, a_{m+2}$ *are those numbers, which determine the hyperplane* A, $A \in \mathcal{K}$.

Proof. Let τ be an efficient stopping time. Thus the equality (3) is satisfied for some $a_1, a_2, \ldots, a_{m+2}$. Suppose that the hyperplane A, given by these numbers does not belong to $\mathcal{K}$. According to (4), for each $\theta \in \Theta_1$ there exists $\delta > 0$ such that

$$P_\theta\{Z(\eta_{a_{m+2}}) > a_{m+2} + \delta\} > 0. \tag{5}$$

The process $\{Z(t)\}_{t\in T}$ has also stationary independent increments and it satisfies the strong Markov property for η_s [16]. Thus it is not difficult to see that

$$P_\theta\{\tau_A = +\infty\} \geq P_\theta\{Z(\eta_{a_{m+2}}) > a_{m+2} + \delta\} P_\theta\{\inf_{\eta_{a_{m+2}} < t < +\infty} (Z(t) - Z(\eta_{a_{m+2}})) > -\delta\}. \tag{6}$$

We shall show that, for each $\theta \in \Theta_1$,

$$P_\theta\{\inf_{\eta_{a_{m+2}} < t < +\infty} (Z(t) - Z(\eta_{a_{m+2}})) > -\delta\} > 0. \tag{7}$$

Let $\sigma_s := \inf\{t > 0: (-Z(t)) \geq s\}$, $s > 0$. Obviously

$$P_\theta\{\inf_{\eta_{a_{m+2}} < t < +\infty} (Z(t) - Z(\eta_{a_{m+2}})) > -\delta\} = P_\theta\{\sup_{0<t<+\infty} (-Z(t)) < \delta\},$$

$$P_\theta\{\sup_{0<t<+\infty} (-Z(t)) < \delta\} \leq P_\theta\{\sigma_\delta = +\infty\}, \tag{8}$$

$$P_\theta\{\sup_{0<t<+\infty} (-Z(t)) < \delta\} \geq P_\theta\{\sigma_{\delta_1} = +\infty\}, \quad 0 < \delta_1 < \delta.$$

Using the fact that $\{Z(t)\}_{t\in T}$ satisfies the strong Markov property for σ_s [16], one can easily get:

$$\begin{aligned} P_\theta\{\sigma_{2s} < +\infty\} &\geq P_\theta\{\sigma_s < +\infty\} P_\theta\{\inf(t > \sigma_s: -Z(t) + Z(\sigma_s) \geq s) < +\infty\} \\ &= P_\theta\{\sigma_s < +\infty\} P_\theta\{\sigma_s < +\infty\}, \quad s > 0. \end{aligned}$$

Suppose now that $P_\theta\{\sigma_{\delta_1} < +\infty\} = 1$, $\theta \in \Theta_1$. Thus $P_\theta\{\sigma_{2\delta_1} < +\infty\} = 1$, which yields $P_\theta\{\sigma_s < +\infty\} = 1$ for every $s > 0$. Thus in accordance with (8) we obtain

$$P_\theta\{\sup_{0<t<+\infty} -Z(t) < +\infty\} = 0, \quad \theta \in \Theta_1. \tag{9}$$

But since $E_\theta(-Z(1)) < 0$, $\theta \in \Theta_1$, we have that [9, p. 494]

$$P_\theta\{\sup_{0<t<+\infty} -Z(t) < +\infty\} = 1, \quad \theta \in \Theta_1,$$

which contradicts (9). Thus $P_\theta\{\sigma_{\delta_1} < +\infty\} < 1$, which with (8) yields (7). From inequalities (5), (6) and (7) we get

$$P_\theta\{\tau_A=+\infty\}>0, \quad \theta\in\Theta_1$$

which with the obvious fact $\tau\geq\tau_A$ yields

$$P_\theta\{\tau=+\infty\}>0, \quad \theta\in\Theta_1.$$

Since we consider only finite stopping times, we obtain that if the stopping time τ is efficient, then the measure generated by $(S(\tau),\tau)$ is accumulated on a hyperplane which belongs to $\mathscr{H}$. Suppose that the stopping time τ is efficient. Then relation (3) is satisfied. Let τ_A be its corresponding time of the first passage. According to the efficiency of τ and the fact $\tau\geq\tau_A$ we have $E_\theta\tau<+\infty$, $E_\theta\tau_A<+\infty$, and according to [4, 12],

$$E_\theta Z(\tau)=E_\theta\tau E_\theta Z(1), \quad E_\theta Z(\tau_A)=E_\theta\tau_A E_\theta Z(1), \qquad \theta\in I.$$

Hence $E_\theta\tau=E_\theta\tau_A=a_{m+2}/E_\theta Z(1)$, which yields $\tau=\tau_A$ P_θ-a.s. for each $\theta\in I$. Thus the efficient stopping times belong to W. We shall show that the inequalities in (2) are satisfied for each $\tau_A\in W$ and $\theta\in\Theta_1$. Taking into account Remark 2, we have $\tau_A=\eta_{a_{m+2}}P_\theta$-a.s. for each $\theta\in\Theta_1$, when $\tau_A\in W$. From the results of A. Gut [10, 11] concerning the finiteness of the moments of first passage times, we have $E_\theta\eta^2_{a_{m+2}}<+\infty$ for each $\theta\in\Theta_1$. The remaining inequalities in (2) follow from the results in [4, 12]. According to (1) we have that every efficient estimator can be represented as a linear function of the components of the sufficient statistic. To complete the proof of Theorem 1 we shall show that every linear function of $S_1(\tau_A),\ldots,S_m(\tau_A)$, τ_A, can be represented in the form given in (1) for each $\theta\in\Theta_1$, $\tau_A\in W$. Introduce the following notation:

$$U_i:=S_i(\tau_A)+\frac{\partial\beta}{\partial\theta_i}\tau_A, \quad i=1,2,\ldots,m. \tag{10}$$

Of course

$$\sum_{i=1}^m a_iU_i=\sum_{i=1}^m a_iS_i(\tau_A)+\tau_A\sum_{i=1}^m a_i\frac{\partial\beta}{\partial\theta_i},$$

and according to (3) we have

$$\sum_{i=1}^m a_iU_i=a_{m+2}-\tau_A\left(a_{m+1}-\sum_{i=1}^m a_i\frac{\partial\beta}{\partial\theta_i}\right).$$

But as one can see, $E_\theta Z(1)=a_{m+1}-\sum_{i=1}^m a_i(\partial\beta/\partial\theta_i)$. Since $E_\theta Z(1)>0$ for each $\theta\in\Theta_1$, we obtain that τ_A can be represented as a linear function of U_i, $i=1,2,\ldots,m$, and from (10) the same is true for each $S_i(\tau_A)$, $i=1,2,\ldots,m$. Using relation (3) and Wald's equations,

$$E_\theta S_i(\tau_A)=-\frac{\partial\beta}{\partial\theta_i}E_\theta\tau_A, \quad \theta\in\Theta_1,$$

and we immediately get all efficiently estimable functions. □

4. Applications

From a practical point of view it is important to know how one can construct the hyperplanes from $\mathcal{K}$. Thus in this section we obtain useful expressions (11), (12) and (13) for determining the explicit form of the hyperplanes from $\mathcal{K}$. Some examples are also given.

Consider first the case when $\{S(t)\}_{t\in T}$ is a discrete time process. We shall suppose that the distribution of $S(1)$ (i) is discrete, (ii) is absolutely continuous with respect to the Lebesgue measure, where the density function is continuous on the domain where it is positive, or, (iii) consists of these two parts mentioned in (i) and (ii). Let τ_A belong to W and let $Z(t)$ be defined as above. Since $E_\theta Z(1)>0$ $\theta\in\Theta_1$, we have one of the following three cases:

1. There exist $h_1>0$ and $h_2>0$, $h_1\neq h_2$, such that $P_\theta\{Z(1)=h_i\}>0$, $i=1,2$.

2. There exists an interval $[a, b]\subset(0,+\infty)$, such that the distribution of $Z(1)$ is absolutely continuous on $[a, b]$ with respect to the Lebesque measure and the density function is positive on $[a, b]$.

3. There exists $h>0$, such that $P_\theta\{Z(1)=h\}>0$ and $P_\theta\{Z(1)\in(-\infty,0]\}=1-P_\theta\{Z(1)=h\}$.

Case 1. Let $h_1<h_2$. It is easy to see that for each $s>0$ there exists a natural κ such that

$$P_\theta\{Z(\eta_s)>s\}\geq P_\theta^\kappa\{Z(1)=h_1\}P_\theta\{Z(1)=h_2\}>0,\quad \theta\in\Theta_1.$$

Case 2. It is easy to see that for each $s>0$, there exist a suitable subinterval $[x, y]\subset[a, b]$ and a natural κ such that

$$P_\theta\{Z(\eta_s)>s\}\geq P_\theta^\kappa\{Z(1)\in[x, y]\}>0,\quad \theta\in\Theta_1.$$

Thus if $\tau_A\in W$, then the distribution of $Z(1)$ is of the type given in Case 3. Furthermore, in a similar way to the above, it is easy to show that the equality $P_\theta\{Z(\eta_s)>s\}=0$, $\theta\in\Theta_1$, holds for some $s>0$ if and only if

$$P_\theta\{Z(1)\in(h, 0, -h, -2h, \ldots)\}=1,\quad \theta\in\Theta_1. \tag{11}$$

Of course s may take only the values h, $2h$, $3h, \ldots$.

Consider now the case when $\{S(t)\}_{t\in T}$ is a continuous time jump process. Let t_s denote the moment of the sth jump of the process. We suppose that the distribution of $S(t_1)$ is of the same type as that of $S(1)$ in the discrete time case considered above. Let $\tau_A\in W$ and $Z(t)$ be defined as above. Consider the case when $a_{m+1}\neq 0$. In a similar way to the above for the discrete time process, it is not difficult to show that the equality $P_\theta\{Z(\eta_s)>s\}=0$ holds for some $s>0$ if and only if

$$\alpha_{m+1}>0,\qquad P_\theta\{\sum_{i=1}^m a_iS_i(t_1)>0\}=0,\quad \theta\in\Theta_1. \tag{12}$$

The case $a_{m+1}=0$ leads to the discrete time case. Thus in this case we have that

$A \in \mathcal{K}$ if and only if

$$P_\theta\{Z(t_1) \in (h, 0, -h, -2h, \ldots)\} = 1, \quad h > 0, a_{m+2} = h, 2h, \ldots, \theta \in \Theta_1 \tag{13}$$

Consider now the case when $\{S(t)\}_{t\in T}$ is a continuous time process, whose trajectories are continuous. Of course each hyperplane A belongs to $\mathcal{K}$.

Example 1 (Multinomial process [3, 17]). A random process $\{X(t)\}_{t\in T}$, $T = \{0, 1, 2, \ldots\}$, with values in $\mathbb{R}^m$ and with stationary independent increments, is said to be multinomial if $X(0) = 0$ and

$$P\{X(1) = e_i\} = p_i, \quad i = 1, 2, \ldots, m, \tag{14}$$

where

$$0 < p_i < 1, \quad i = 1, 2, \ldots, m, \qquad \sum_{i=1}^{m} p_i = 1$$

and

$$e_1 = (1, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \ldots, \quad e_m = (0, \ldots, 0, 1).$$

The unknown parameter is $p = (p_1, p_2, \ldots, p_{m-1})$. Of course the process $(X_1(t), \ldots, X_{m-1}(t))_{t\in T}$ belongs to the exponential class [24] and the equality (3) takes the following form (observing that $t = \sum_{i=1}^{m} X_i(t)$):

$$\textstyle\sum_{i=1}^{m} a_i X_i(\tau) = a_{m+2},$$

where $a_{m+2} > 0$, $\sum_{i=1}^{m} a_i^2 \neq 0$. Taking into account (11), without loss of generality we can assume that $a_1, a_2, \ldots, a_m, a_{m+2}$ are integers and their greatest common divisor equals 1. Then immediately from (11) we obtain that the hyperplane A belongs to $\mathcal{K}$ if and only if $a_i \leq 1$ for each i, $i = 1, 2, \ldots, m$, where at least for one i_0, $1 \leq i_0 \leq m$, $a_{i_0} = 1$.

Example 2 (Compound Poisson process with exponential jump [18]). Consider the process

$$X(t) = \sum_{i=1}^{N(t)} \xi_i$$

where $\xi_1, \xi_2, \ldots$ are independent exponentially distributed random variables with parameter μ, $\mu \in (0, +\infty)$, independent of the process $\{N(t)\}_{\in T}$. Here $\{N(t)\}_{t\in T}$ is a classical Poisson process with parameter λ, $\lambda \in (0, +\infty)$. The unknown parameter is (λ, μ). The process $\{X(t), N(t)\}_{t\in T}$ forms a sufficient statistic which belongs to the exponential class [18]. The equality (3) takes the form

$$a_1 X(\tau) + a_2 N(\tau) + a_3 \tau = a_4$$

where $a_4 > 0$, $a_1^2 + a_2^2 + a_3^2 \neq 0$. Immediately from (12) and (13) we obtain that the hyperplane A belongs to $\mathcal{K}$ if and only if one of the following two conditions is fulfilled:

1. $a_3 > 0,\ a_1 \leqslant 0,\ a_2 \leqslant 0$ (see (12)),
2. $a_3 = 0,\ a_1 = 0,\ a_2 > 0,\ a_4 = a_2, 2a_2, \ldots$ (see (13)).

Example 3 (Compound Poisson process with multinomial jumps). Consider a compound Poisson process

$$X(t) = \sum_{i=1}^{N(t)} \xi_i$$

where the jump ξ_i has the multinomial distribution (14). The unknown parameter is $(\lambda, p_1, p_2, \ldots, p_{m-1})$. It is easy to see that the process $(X_1(t), \ldots, X_m(t))_{t \in T}$ forms a sufficient statistic which belongs to the exponential class (observe that $N(t) = \sum_{i=1}^{m} X_i(t)$). The equality (3) takes the form

$$\sum_{i=1}^{m} a_i X_i(\tau) + a_{m+1}\tau = a_{m+2}$$

where $a_{m+2} > 0$, $\sum_{i=1}^{m+1} a_i^2 \neq 0$. Immediately from (12) and (13) we have that the hyperplane A belongs to $\mathscr{H}$ if and only if one of the following two conditions is satisfied:

1. $a_{m+1} > 0,\ a_i \leqslant 0,\ i = 1, 2, \ldots, m$ (see (12)),
2. $a_{m+1} = 0,\ a_1, a_2, \ldots, a_m, a_{m+2}$ are integers and $a_i \leqslant 1$ for each i, $i = 1, 2, \ldots, m$, where at least for one i_0, $1 \leqslant i_0 \leqslant m$, $a_{i_0} = 1$ (see (13)).

Acknowledgement

The author is indebted to the referee whose comments improved the presentation of the paper.

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